uwait University — Math 101 — May 10, 2001
[ath. & Comp. Sci. Dept. — Second Exam — Time: 75 min.

Calculators and Mobile Phones are not allowed.

1. (2 Points) Use the definition of the derivative to find $f'(1)$ if $f(x) = \sqrt[3]{x}$.

2. (2 Points) Use differentials to find an approximate value to $\sqrt{(2.02)^3 + 1}$.

3. (3 Points) Let $f(x) = ax^2 - 12x + 8$. Find all values of a such that the tangent line to the graph of f at $x = 3$ is parallel to the line $y - 6x + 1 = 0$.

4. (3 Points)Find $\dfrac{dy}{dx}$ at $x = 0$ if $xy + (x+y)^3 = 1$.

5. (3 Points) Find $f'(t)$ if $f(t) = \tan(\sqrt{t^2+1}) + t(t^2+1)^5$.

6. (3 Points)

   a) State the Mean Value Theorem.

   b) Use the mean value theorem to show that

$$(1+x)^{\frac{2}{3}} < 1 + \tfrac{2}{3}x, \qquad x > 0.$$

7. (3 Points) A box of a rectangular base and an open top has surface area of 600 cm^2. If the height of the box is equal to its width, find the dimensions that give the box a maximum volume.

8. (6 Points) Let $f(x) = \dfrac{1-x^2}{x^2}$.

   a) Find the vertical and horizontal asymptotes for the graph of f (if any).

   b) Find the intervals on which f is increasing and the intervals on which f is decreasing, and find the local extrema of f (if any).

   c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward, and find the points of inflection (if any).

   d) Is the graph of f symmetric with respect to the y-axis? Justify your answer.

   e) Sketch the graph of f.

By def. $f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{\cdots}{x-1} = \lim_{x\to 1} \frac{\cdots}{(x^{1/3}-1)(x^{2/3}+x^{1/3}+1)}$

$= \boxed{\frac{1}{3}}$.

Let $f(x) = \sqrt{x^3+1}$, then $f(2) = 3$, $\Delta x = 0.02$, $f'(x) = \frac{3x^2}{2\sqrt{x^3+1}} \Rightarrow f'(2) = 2$

since $f(x+\Delta x) \simeq f(x) + f'(x)\cdot \Delta x \Rightarrow$

$f(2+0.02) = \sqrt{(2.02)^3+1} \simeq f(2) + f'(2)\cdot(0.02) = 3 + 2(0.02) = \boxed{3.04}$.

Slope of line is 6, and $f'(x) = 2ax - 12$. Since the tangent line to the graph of f at the point whose x-coordinate is $3 \Rightarrow$

$f'(3) = 2a(3) - 12 = 6 \Rightarrow 6a = 18 \Rightarrow \boxed{a = 3}$.

4. Using I.D. $\Rightarrow$ $y + xy' + 3(x+y)(1+y') = 0$. If $x = 0 \Rightarrow y^3 = 1 \Rightarrow y = 1$

and $1 + 3(1)(1+y'(0)) = 0 \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = \boxed{-\frac{4}{3}}$.

$f'(t) = \sec^2(\sqrt{t^2+1})\, \frac{2t}{2\sqrt{t^2+1}} + (t^2+1)^5 + 5t(t^2+1)^4(2t)$

$= \frac{t}{\sqrt{t^2+1}} \sec^2(\sqrt{t^2+1}) + 10t^2(t^2+1)^4$.

b) Let $f(x) = (1+x)^{2/3}$ on $[0,x]$, $x > 0$. Since f is conts on $[0,x]$, diff. on $(0,x) \Rightarrow$ by MVT $\Rightarrow$

$f'(c) = \frac{(1+x)^{2/3} - 1}{x} \Rightarrow \frac{2}{3}\frac{1}{\sqrt[3]{1+c}} = \frac{(1+x)^{2/3} - 1}{x}$

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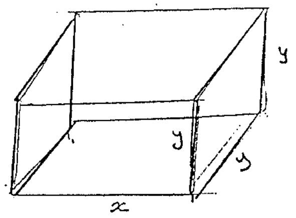
$\Rightarrow (1+x)^{2/3} = 1 + \frac{2}{3}\frac{x}{\sqrt[3]{1+c}}$, $c \in (0,x)$, $x > 0$ and since $\frac{1}{\sqrt[3]{1+c}} < 1$

$\Rightarrow (1+x)^{2/3} < \frac{2}{3}x$, $x > 0$.

Surface Area $= xy + 2y^2 + 2xy = 600$ (1)

Volume $V = (xy)(y) = xy^2$ (2)

From (1) $\Rightarrow$ $x = \frac{600 - 2y^2}{3y}$ (1')



Using (1') in (2) $\Rightarrow$

$V = \left(\frac{600-2y^2}{3y}\right) y^2 = \frac{1}{3}(600y - 2y^3)$, $D_V = [0, 10\sqrt{3}]$.

$V''(y) < 0 \Rightarrow V$ has a local max. at $y = 10$

The dimensions of the box are: $x = \frac{600-200}{30} = \boxed{\frac{40}{3}}$ cm

$y = \boxed{10}$ cm, height $= \boxed{10}$ cm.

3. $f(x) = \frac{1-x^2}{x^2} = \frac{1}{x^2} - 1$, $D_f = \mathbb{R}/\{0\}$

a) For V.A: $\boxed{x=0}$ is a V.A., $\lim_{x\to 0^-} \frac{1-x^2}{x^2} = \infty$, $\lim_{x\to 0^+} \frac{1-x^2}{x^2} = \infty$

H.A: $\lim_{x\to+\infty} \frac{1-x^2}{x^2} = -1 \Rightarrow \boxed{y=1}$ is a H.A.

$\lim_{x\to-\infty} \frac{1-x^2}{x^2} = -1$

4.
b) $f'(x) = -\frac{2}{x^3} \Rightarrow f'(x)$ DNE if $x = 0$

Interval	$(-\infty,0)$	$(0,\infty)$
Test #	-1	1
Sign of f'	$+$	$-$
Conclusion	f is Inc. on $(-\infty,0)$	f is Dec. on $(0,\infty)$

f is increasing on $(-\infty,0)$; decreasing on $(0,\infty)$. No local extrema of f.

c) For Concavity: $f''(x) = \frac{6}{x^4} > 0$, $\forall x$, $x \neq 0 \Rightarrow$ The graph of f is CU.

No inflection points.

8 d) Since $y = f(x) = \frac{1-x^2}{x^2}$, then $f(-x) = f(x)$ and the graph of f is symmetric w.r. to the y-axis.

e) Help points: x-intercept: $\frac{1-x^2}{x^2} = 0 \Rightarrow x = \pm 1 \Rightarrow (-1,0), (1,0)$ on the graph of f. No y-intercept since $x = 0 \notin D_f$.